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Analytic Solutions for Asymmetric Model of a Rod in a Lattice Fluid

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We consider the problem of determining analytically some exact solutions of the concentration $u(x, y, t)$ of particles moving by diffusion and advection or drift. It is assumed that the advection is nonlinear. The driven diffusive flow is impeded by an impenetrable obstacle (rod) of length L. The exact solutions for u are evaluated for small and big values of vL/D, where v is the drift velocity and D is the diffusion coefficient. The results show that in some regions in the (x, y) plane the concentration first increases (or decreases) monotonically and then is nearly constant after some critical length L. The location at which u is nearly constant depends on the nature of the driving field v/D. This problem has relevance for the size segregation of particulate matter which results from the relative motion of different-size particles induced by shaking. Methods of symmetry reduction are used in solving the nonlinear advection-diffusion equation in $(2+1)$ dimensions.

KEY WORDS: Lattice fluid models; advection-diffusion processes; symmetry reduction method.

I. INTRODUCTION AND SURVEY

Size segregation effects are important in many industrial situations in which granular mixtures of particles of different sizes are used. Because of its industrial importance in powder metallurgy, pharmaceuticals and the glass and paint industries, it has been the subject of considerable study in the engineering community.[1–5] The emphasis in these studies has been on the segregation rate and its independence on parameters such as particle size and weight ratio and shaking frequency, rather than analytical description of this phenomena. In an effort to better understand the dynamics of the segregation process, the dynamical picture of this phenomena is

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modeled by Rosato, Strandburg, Prinz and Swendsen (RSPS).[6, 7] They conducted a series of computer simulations on mixtures exhibiting such a segregation. Their simulations gave results consistent with experiments; after many shakes, the larger disks lie on top of the smaller ones in a non-equilibrium stationary state. Motivated by their work, Alexander and Lebowitz (AL)[8, 9] investigated the behaviour of a simple lattice gas system, which exhibit a similar size-dependent relative motion, to model segregation of different sized particles subjected to shaking in a gravitational field. The model consists of a gas of monomers and a single rod of length L on a two-dimensional square lattice, Z^2, and that the rod is rigidly aligned in the vertical direction. A monomer occupies one site and the rod more than one. The particles interact by exclusion; no more than one particle per site is permitted A monomer at site x waits an exponential time with parameter one and then selects a lattice direction $\underset{\sim}{e}=(e_1, e_2)$ with probability $P(\underset{\sim}{e})$. If the neighbouring site y in the direction $\underset{\sim}{e}$; $y=x+\underset{\sim}{e}$ is unoccupied at that time, the particle jumps to that site; otherwise it does not move. The rod moves according to a dynamics similar to that of the monomers; the rod of length L to move horizontally, all L sites immediately adjacent to it in the direction of motion must be simultaneously unoccupied, and the vertical motion requires only that the one site adjacent in that direction be empty. When there is no rod present, this model is just the much studied simple exclusion process,[10] and if the exchange rates are symmetric; $P(\pm e_1)=P(\pm e_2)=1/4$, the process is reversible and under a proper resealing of space and time the rod motion converges to standard Brownian motion with a positive diffusion constant. When the exchange rates are not symmetric, the process is non-reversible and the particles attempted to move to a neighbouring sites in one particular direction by the probabilities $P(\underset{\sim}{e})$, for example: if $e_1=\text{east}$, $e_2=\text{north}$

$$P(-e_1)=0, \qquad P(e_1)=a, \qquad P(-e_2)=P(e_2)=b=(1-a)/2$$

To find the behaviour of this system, (AL)[9] carried out a comprehensive computer simulations Which showed a surprising relationship between the rod's velocity and its length in the stationary state; that beyond a certain length, the longer rods moved faster, although more sites need to be empty in order for the longer rods to move. The anomalous behaviour of the rod velocity led them to study the probability that all of the sites immediately to the right of the rod were simultaneously unoccupied. Simulations have showed that the long rods, whether stationary or moving, distort the local monomer profile to a state which is independent of the monomer density and create a larger depletion region to the right of them. In an effort to better understand the asymmetrical interacting particle model with two kinds of particles, (AL)[9] have presented a detailed description of some related continuum models whose behaviour is quite similar to that for the particle model. As one of these models, the continuum problem of driven diffusive flow past an impenetrable obstacle, and the obstacle normal to the incident flow. It is assumed that the flux of particles in the fluid consists of a diffusive part $-D\ u(x, y, t)$ and a linear drift part $v\ u(x, y, t)$; where $u(x, y, t)$ is the particle concentration, D is the diffusion coefficient, and v is the drift velocity. Therefore, when D and v are constant, the equation of continuity gives

$$u_t = D(u_{xx} + u_{yy}) - vu_x \tag{1}$$

where subscript represents partial derivative. In the steady state, Eq. (1) was studied by Phillip *et al.* in ref. 11, they were considering the flow of groundwater around a cylindrical obstacle. They obtained an exact solution in the form of an infinite series. In (AL),[9] they stated that the solution of Phillip *et al.* has the qualitative features of the density profile resemble those observed in the computer simulations of monomer flow past a stationary obstacle. The same problem was considered by Knessl and Keller (KK).[12] They were considering the effect of an impenetrable obstacle upon the concentration of the particles in a fluid when the particles moving by diffusion and linear advection.

The case of primary concern in this paper is to study another model of the driven diffusive flow whose behaviour is quite similar to that of the rod in lattice fluid. Following (AL),[9] we wish to investigate the behaviour of the continuous density of particles moving by diffusive flux perpendicular to the (nonlinear) drift, under the condition that there can be no flux through an obstacle of length L centered on the origin and oriented perpendicular to the drift.

In the next section a formulation of the problem is summarized in a form convenient for later use. Then, a two-parameter family of exact closed-form solutions to non-linear advection-diffusion equation in $(1+2)$ dimension, which governs the problem, is presented and corresponding evolution of the probability density is discussed. In the appendix we describe briefly the symmetry group analysis of the partial differential equation which enables us to obtain these great variety of solutions.

II. FORMULATION

Consider the continuum problem of driven diffusive flow of particles past an impenetrable obstacle in two dimensions see Fig. 1. The obstacle is an impenetrable strip of length L and is parallel to the z-axis; the strip is

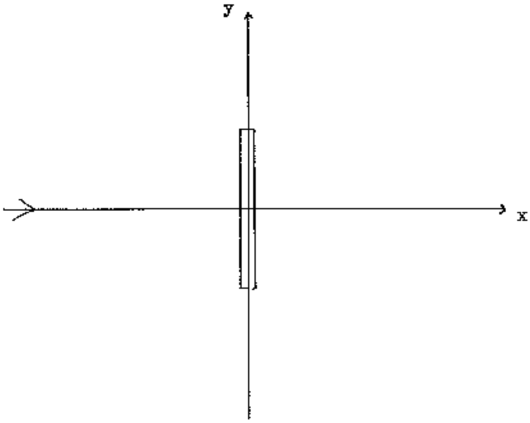

Fig. 1. A strip parallel to the z axis of length L. The strip intersects the (x, y) plane along the interval $y = -L/2$ to $y = L/2$ on the axis. The direction of advection is parallel to the x axis from left to right. The moving frame of the strip.

normal to the incident flow. The flux of particles in the fluid is composed of row tends: a linear diffusive term $-Du_y$ perpendicular to the nonlinear drift tend $vLu(1-u)$, where $u(x, y, t)$ is the particle concentration, D is the diffusion coefficient, and v is the drift velocity. The velocity v may result from motion of the fluid (advection), or from gravitational field acting on the particles (drift). Conservation of particles implies that the divergence of the flux equal to $-u_t$. Therefore when D and v are constant, $u(x, y, t)$ satisfies the partial differential equation,

$$u_t = Du_{yy} - vL(1-2u)\,u_x \tag{2}$$

In the steady state, Eq. (2) becomes

$$u_{yy} = \frac{vL}{D}(1-2u)\,u_x \tag{3}$$

Since u is the particle concentration, a solution of physical interest should be non-negative and bounded.

Physically interesting problems usually have some symmetries. Using these symmetries; we can simplify Eq. (2) to a certain extent. We used Lie group analysis to find all the invariants of the symmetries, and constructing solutions with these invariants. The similarity transformation method based on Lie group analysis has many applications when dealing with differential equations and related physical problems.[13–15] Especially in the nonlinear case, it can sometimes help us in finding physically meaningful exact solutions. The equation (2) we are going to consider is strongly non-linear, and it is desirable and interesting to find all analytic solutions, see appendix. One purpose of this study is to sketch the feature of each type of the solutions.

III. QUALITATIVE FEATURES OF SOME EXACT SOLUTIONS

In the following we shall consider some types of our exact solutions of Eq. (2), which determine the concentration of the particles at any region in (x, y) plane. The explicit form of u contains the obstacle length L and drift velocity v, and the diffusion coefficient D, so we are able to discuss the behaviour of u as a function of them. The exact solution contains two arbitrary constants k and C, which can be chosen so that u simulates some desired physical situation, or the initial distribution $u(x, y, o)$ has some desired features, which mean a great variational in the solution.

1. The First Type Solution

Let us consider time-dependent solution of Eq. (2),

$$u(x, y, t) = \left(\frac{1}{2} - \frac{k}{2vL}\right)\left\{1 + \tanh\left[\left(\frac{vL - k}{2D}\right)(x + y - kt) + C\right]\right\} \qquad (4)$$

The concentration of the particles in the stationary state, can be derived from Eq. (4) by letting $k = 0$,

$$u(x, y) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{vL}{2D}(x + y) + C\right] \qquad (5)$$

which is the exact solution of Eq. (3).

The constant C affects the solution u mainly as a scaling parameter in (x, y)-plane, i.e., we can chose C big or small enough to satisfy that the concentration $u(x, y)$ have the uniform value unity. Taking into account the properties of the hyper-tangent function, one can see that the concentration of particles is bounded and non-negative as time tends to infinity at any point in (x, y)-plane, i.e., $0 \leqslant u(x, y) \leqslant 1$, and it reaches maximum about $(vL/2D)(x + y) \approx 2.8$, minimum about $(vL/2D)(x + y) \approx -2.8$, and $u = 1/2$ at $(vL/2D)(x + y) = 0$. These indicate that the presence of an obstacle, whether stationary or moving, in a driven diffusive flow with non-linear drift will distort the local concentration profile to a state which divided the (x, y)-plane into two regions, not as expected about the strip axis $x = 0$, but about the straight line $x + y = 0$. The concentration is relatively higher in one side than the other side, apart from the value of vL/D.

If we look at the concentration $u(L)$ as a function of the obstacle length it appears that, the concentration will increase (or decrease) monotonically and then it approach a constant value after certain length L_o, depending on the indicated regions in the (x, y)-plane.

Actually, the concentration profile has not any difference when the drift is weak or strong relative to the diffusion; v/D may affects the solution u mainly as a catalytic or inhibitory agent to achieve its maximum or minimum. The typical behaviour of solution (5), with $C=o$ is plotted in Fig. 2 using MAPLE.

2. The Second Type Solution

In the following we present three forms of the time-dependent solution of Eq. (2), the corresponding stationary solutions can be obtained by letting the arbitrary constant $k=0$;

$$u(x, y, t)=\frac{1}{2}-\frac{k}{2vL}+\frac{2(kt-x)}{vL(C+\sqrt{2/3D}\ y)^2} \tag{6}$$

the corresponding stationary solution is

$$u(x, y)=\frac{1}{2}-\frac{2x}{vL(C+\sqrt{2/3D}\ y)^2} \tag{7}$$

The second form is,

$$u(x, y, t)=\frac{1}{2}-\frac{k}{2vL}+\frac{3D(kt-x)}{vL(y^2-6CD(kt-x))^{4/3}} \tag{8}$$

and the stationary solution is

$$u(x, y)=\frac{1}{2}-\frac{x}{vL(y^2-6CDx^{4/3})} \tag{9}$$

the last formal solution is,

$$u(x, y, t)=\frac{1}{2}-\frac{k}{2vL}+\frac{(kt-x)+C}{vLy^2/3D} \tag{10}$$

where the corresponding stationary solution is

$$u(x, y)=\frac{1}{2}+\frac{C-x}{vLy^2/3D} \tag{11}$$

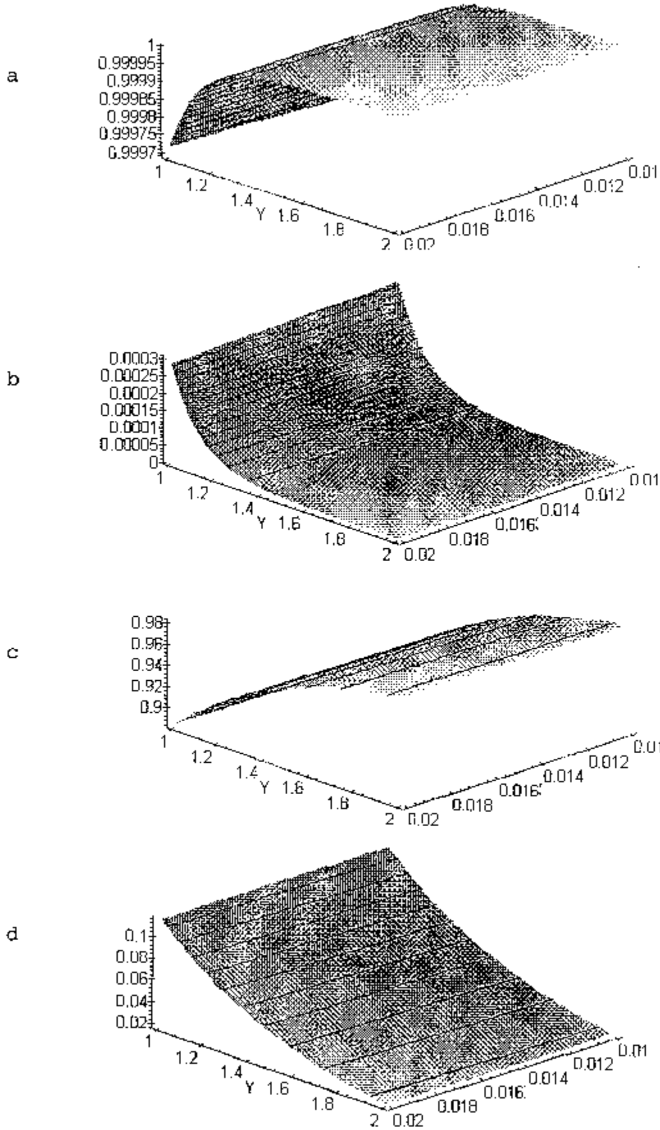

Fig. 2. The hyper-tangent solution (5) of Eq. (3), with $C=0$ where (a) $vL/D=8$ in region $x+y>0$ (b) $vL/D=8$ in region $x+y<0$, (c) $vL/D=2$ in region $x+y>0$ (d) $vL/D=2$ in region $x+y<0$. This plot was drawn using MAPLE.

Further types of exact solution are obtained in the appendix. Inspection of the stationary solutions (7), (9) and (11), with arbitrary constant $C = O$, reveals that the concentration of particles in (x, y) plane has the same form

$$u(x, y) = \frac{1}{2} - \frac{3Dx}{vLy^2} \tag{12}$$

and it changes sign when x changes sign. This indicates that the density u will be non-uniform and as we expect the obstacle to block the flow and to produce an accumulation of particles on its front side; when $x = 0^-$, and a depletion of particles on its back side; $x = 0^+$, It appears that the concentration $u(x, y)$ has symmetry $u(x, y) = u(x, -y)$, but $u(x, y) + u(-x, y) = 1$.

To have an explicit solution of Eq. (3), this enables one to discuss the effect of the obstacle length L to the concentration u; which shows that the concentration has inverse proportional with L, i.e., in front of the obstacle, the accumulation of particles decrease monotonically with L, but increase monotonically with L in the back side of the obstacle. Similarly, the effect of the quantity v/D upon the particle concentration looks like L. The point of considerable interest is that, the concentration of particles has a parabolic cylinder profile in the front side of the obstacle, where $x < o$, and in back side where $x > o$. This can be explained as follows. Consider the regions in (x, y)-plane in which the concentration u has some fixed value; say $u = u_o$, then in both sides, we have $y^2 = Ax$, where $A = 3D/vL(u_o - 1/2)$, which is the standard canonical equation of the parabola, its axis coincides with x-axis and A is the parameter of the parabola. The smaller the absolute-value of A, the closer the focus to the vertex and the more spread out the parabola is. The parabolic curve in the xy-plane represents in (u, x, y)-space a cylindrical surface whose generatrix is parallel to the u-axis and the line of the parabola is the directrix. In view of this analysis, there is a wing-like structure in the concentration pattern of the particles, which agrees with that observed by (AL).[9] Plots of the solution (12) for different values of VL/D in the two regions are given in Fig. 3, these plots were drawn using MAPLE.

IV. DISCUSSION AND CONCLUDING REMARKS

One of our motivations for studying this system was to provide some exact analytic solutions of the non-linear advection-diffusion equation in $(1 + 2)$-dimension (2), which describes well the concentration of particles in the presence of an obstacle in driven system. We have derived a number of exact solutions of different variety, which contain parameters and constants

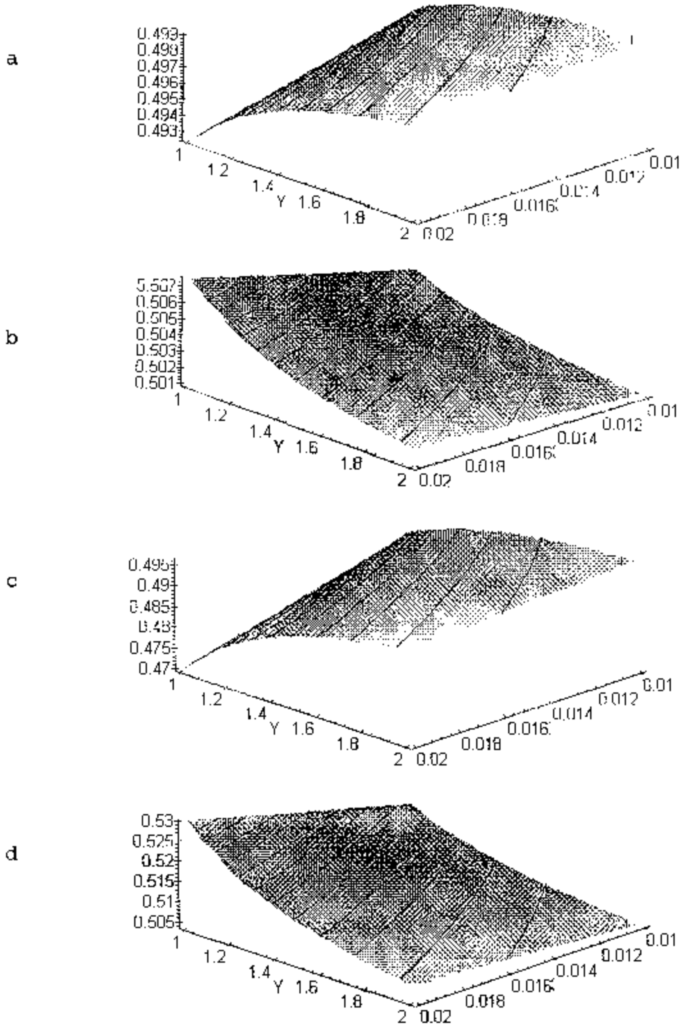

Fig. 3. The parabola-solution (12) of Eq. (3) where (a) $vL/D = 8$ in region $x > 0$ (b) $vL/D = 8$ in region $x < 0$, (c) $vL/D = 2$ in region $x > 0$ and (d) $vL/D = 2$ in region $x < 0$ and the contour lines of parabola shape.

can be chosen to simulate some desired physical situations. The time-dependent solutions contain terms $(kt - x)$ or $(x + y - kt)$ which indicate that the particles will accumulate in a wave-like feature. The stationary solutions can be obtained by letting $k = O$, and we have two kinds of solutions, one is the hyper-tangent-solution and the other is parabola-solution plotted in Figs. 2 and 3. The plots illustrate connections between the concentration u and the parameter vL/D, and show that the two kinds of solutions have the same profile; consist of two sheets, one of them has a relatively high concentration than the other. The total concentration on the two sheets are the uniform concentration, $u=1$. The difference in the concentration value is clear in regions directly near to the obstacle surface, where $x=0^-$ or $x=0^+$ and after that it will approach the same values. The plots show also that, the longer the obstacle, the big difference in concentration values around the obstacle surface, and create a larger depletion region in a wing-like shaper see contour lines in Fig. 3.

It is hoped that such analysis give some explanation of the anomalous behaviour of this system which observed by (AL),[9] and to provide a quantitative and qualitative understanding of the density profile.

APPENDIX. SYMMETRY GROUP ANALYSIS OF EQ. (2) AND ITS EXACT SOLUTIONS

Group-invariance under infinitesimal transformations is used to generate a wide class of solutions of the non-linear advection-diffusion equation in $(1+2)$ dimension. The partial differential equation in three variables is reduced to an ordinary differential equation. Only self-similar types of solutions are discussed. Though variable-separable types can also be obtained as a type, these are excluded from consideration.

A short resume of the ideas in the technique, as it applies to the given Eq. (2), is first given. Invariance of the differential operator and of the solution surface are the two key requirements.[13–15] Further, consideration is limited here to invariance under infinitesimal transformations. For more general considerations, see ref. 14. Consider the transformations

$$\begin{aligned} \underline{x} &= x + \varepsilon X(x, y, t) + 0(\varepsilon^2), \qquad \underline{y} = y + \varepsilon Y(x, y, t) + 0(\varepsilon^2) \\ \underline{t} &= t + \varepsilon T(x, y, t) + 0(\varepsilon^2), \qquad \underline{u} = u + \varepsilon U(x, y, t, u) + 0(\varepsilon^2) \end{aligned} \tag{A.1}$$

where ε is a small parameter defining the group. By retaining terms of order up to ε only, one can see that the derivatives transform as

$$\begin{aligned} \underline{u}_{\underline{t}} &= u_t + \varepsilon[U_t + (U_u - T_t)\, u_t - Y_t u_y - X_t u_x] + 0(\varepsilon^2) \\ \underline{u}_{\underline{x}} &= u_x + \varepsilon[U_x + (U_u - X_x)\, u_x - Y_x u_y - T_x u_t] + 0(\varepsilon^2) \\ \underline{u}_{\underline{y}} &= u_y + \varepsilon[U_y + (U_u - Y_x)\, u_y - X_y u_x - T_y u_t] + 0(\varepsilon^2) \\ \underline{u}_{\underline{yy}} &= u_{yy} + \varepsilon[U_{yy} + (2U_{uy} - Y_{yy})\, u_y - X_{yy} u_x \\ &\quad - T_{yy} u_t + U_{uu} u_y^2 + (U_u - 2Y_y)\, u_{yy} - 2X_y u_{xy} - 2T_y u_{ty}] + 0(\varepsilon^2) \end{aligned} \tag{A.2}$$

It is, of course, possible to assume that X, Y, T also depend on u; the expressions in (A.2) get more lengthy. However, it can finally be seen that such a dependence drops out. The first requirement of invariance implies that $\underline{u}$ satisfies, as a function of $(\underline{x}, \underline{y}, \underline{t})$, the same differential equation (2) as u. Substitute for the variables and derivatives from (A.2), terms of order ε must vanish. Then equate the coefficients of different derivatives of u to zero. These equations provide constraints in the form of partial differential equations for the determination of X, Y, T and U, yielding

$$\begin{aligned} X(x, y, t) &= c_4 x + c_5 t + c_6 \\ Y(x, y, t) &= 1/2 c_2 y + c_1 \\ T(x, y, t) &= c_2 t + c_3 \\ U(x, y, t, u) &= (c_4 - c_2)(u - 1/2) - c_5/2vL \end{aligned} \tag{A.3}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

The second requirement of invariance implies that u, $\underline{u}$ are the same functions of their arguments. This gives a first order partial differential equation as

$$Tu_t + Xu_x + Yu_y = U \tag{A.4}$$

This equation is solved by the use of the method of characteristics, which are given as solutions of any two ordinary differential equations obtained from

$$\frac{dt}{T} = \frac{dx}{X} = \frac{dy}{Y} = \frac{du}{U} \tag{A.5}$$

The general solution of (A.5) involves three constants, two of them; s and z become new independent variables and the third constant; F, plays the role of a new dependent variable. It should be noted that similarity variables s, z and similarity function $F(s, z)$ obtained form (A.5) are quite different to each other according to the choice of the constants values c_i's, $i = 1 \cdots 6$ in (A.3). To find the partial differential equation satisfied by $F(s, z)$, substitute the obtained transformations in Eq. (2). We are able to distinguish three different types:

$$DF_{zz} - (k - vL + 2vLF)\, F_s = 0,$$

$$\text{where} \quad u = F, \quad s = x = kt, \quad \text{and} \quad z = y \tag{A.6}$$

$$k^2DF_{ss} + vL(1-2F)\,F_s - F_z = 0,$$

$$\text{where} \quad u = F, \quad s = k - x, \quad \text{and} \quad z = t \tag{A.7}$$

$$2zDF_{ss} - 2zF_z - 2F + 1 = 0,$$

$$\text{where} \quad u = F - x/2vLt, \quad s = y, \quad \text{and} \quad z = t \tag{A.8}$$

To get the reduced ordinary differential equations, we apply once more the procedure mentioned above to Eqs. (A.6), (A.7) and (A.8).

Let the infinitesimal transformations

$$\underline{s} = s + \varepsilon S(s, z) + 0(\varepsilon),$$

$$\underline{z} = z + \varepsilon Z(s, z) + 0(\varepsilon^2), \qquad \text{and} \qquad \underline{F} = F + \varepsilon f(s, z, F) + 0(\varepsilon^2) \tag{A.9}$$

(be applied to each of Eqs. (A.6), (A.7), and (A.8). Assuming that they are invariant under the transformations (A.9), we get the explicit form of the infinitesimal S, Z and f. From the integral constants of the characteristic equations $(ds/S) = (dz/Z) = (dF/f)$, we have $r(s, z)$ as a new independent variable and $h(r)$ as a new dependent variable. Substituting by r and $h(r)$ into Eq. (A.6), we get the following reduced ordinary differential equations

$$Dh_{rr} + (2vL\,h - vL + k)\,h_r = 0,$$

$$\text{where} \quad F = h(r), \quad \text{and} \quad r = z - a \tag{A.6.1}$$

$$Dh_{rr} - h^2 = 0,$$

$$\text{where} \quad F = \frac{vL-k}{2vL} + \frac{s}{2vL}h, \quad \text{and} \quad r = z \tag{A.6.2}$$

$$h_r - 6D = 0,$$

$$\text{where} \quad F = \frac{vL-k}{2vL} + \frac{h}{2vLz^2}, \quad \text{and} \quad r = s \tag{A.6.3}$$

$$Dh_{rr} + \frac{1}{3}\frac{d(rh^2)}{dr} = 0,$$

$$\text{where} \quad F = \frac{vL-k}{2vL} + \frac{h}{2vLs^{1/3}}, \quad \text{and} \quad r = zs^{-2/3} \tag{A.6.4}$$

Following the same way, we get for Eq. (A.7).

$$k^2Dh_{rr} - 2vLhh_r = 0, \qquad \text{where} \quad F = h(r) \quad \text{and} \quad r = s + vLz \tag{A.7.1}$$

and for Eq. (A.8) we have

$$2Dh_{rr} + rh_r - 2h + 1 = 0, \qquad \text{where} \quad F = h(r) \quad \text{and} \quad r = sz^{-1/2} \tag{A.8.1}$$

Solutions $h(r)$ lead by back substitution to so-called similarity solutions $u(x, y, t)$ of Eq. (2):

Equation (A.6.1) has the solution

$$h(r) = \frac{vL-k}{2vL}\left[1 + \tanh\left(C + \frac{vL-k}{2D}\, r\right)\right]$$

C is arbitrary constant, then Eq. (2) has the solution

$$u(x, y, t) = \frac{vL-k}{2vL}\left[1 + \tanh\left(C + \left(\frac{vL-k}{2D}\right)(x + y - kt)\right)\right] \tag{A.10}$$

Equation (A.6.2) has the solution

$$h(r) = 4/(C + \sqrt{2/3D}\, r)^2$$

then, Eq. (2) has the solution

$$u(x, y, t) = \frac{vl-k}{2vL} + \frac{2(kt-x)}{vL(C + \sqrt{2/3D}\; y)^2} \tag{A.11}$$

Equation (A.6.3) has the solution

$$h(r) = 6Dr + C$$

then Eq. (2) has the solution

$$u(x, y, t) = \frac{vL-k}{2vL} + \frac{3D(kt-x)+C}{vLy^2} \tag{A.12}$$

Equation (A.6.4) has the solution $h(r) = 6D/(r^2 - 6DC)$, C is constant, then Eq. (2) has the solution

$$u(x, y, t) = \frac{vL-k}{2vL} + \frac{3D(kt-x)}{vL(y^2 - 6DC(kt-x)^{4/3})} \tag{A.13}$$

Equation (A.7.1) has the solution,

$$h(r) = k^2D/(k^2DC - vLr)$$

Then Eq. (2) has the solution

$$u(x, y, t) = k^2D/(k^2DC - vL(ky - x + vLt)) \tag{A.14}$$

Equation (A.8.1) has the solution of the form

$$h(r) = 1/2 + {}_1P_1(-1, 1/2, -r^2/4D)$$

where ${}_1P_1$ is the confluent hypergeometric function, by back substitution we get the exact solution of Eq. (2),

$$u(x, y, t) = \frac{1}{2} - \frac{x}{2vLt} + {}_1P_1(-1, 1/2; -y^2/4Dt)$$

which yields the solution of the form

$$u(x, y, t) = \frac{3}{2} + \frac{y^2}{2Dt} - \frac{x}{2vLt} \tag{A.15}$$

REFERENCES

1. W. M. Visscher and M. Bolsterli, *Nature London* **239**:504 (1972).
2. J. Bridgwater, *Powder Technol.* **15**:215 (1976).
3. J. C. Williams, *Powder Technol.* **15**:245 (1976).
4. D. S. Parsons, *Powder Technol.* **13**:269 (1977).
5. J. Bridgwater and W. S. Foo, *Powder Technol.* **41**:147 (1985).
6. A. Rosato, F. Prinz, K. J. Strandburg, and R. H. Swendsen, *Powder Technol.* **49**:59 (1986).
7. A. Rosato, K. J. Strandburg, F. Prinz, and R. H. Swendsen, *Phys. Rev. Lett.* **58**:1038 (1987).
8. F. J. Alexander and J. L. Lebowitz, *J. Phys. A.: Math. Gen.* **23**:L375 (1990).
9. F. J. Alexander and J. L. Lebowitz, *J. Phys. A.: Math. Gen.* **27**:683 (1994).
10. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin, 1991).
11. J. R. Philip, J. H. Knight, and R. T. Waechter, *Water Resources Res.* **25**:16 (1989).
12. C. Knessl and J. B. Keller, *J. Math. Phys.* **38**(1):267 (1997).
13. L. Dresner, *Similarity Solutions of Nonlinear Partial Differential Equations* (Pitman, New York, 1983).
14. P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 1986).
15. L. A. Peletier, *Applications of Nonlinear Analysis in the Physical Sciences*, H. Amann, N. Bazley and K. Kirchgassner, eds. (Pitman, London, 1981).